Asymptotic Behavior of Tails and Quantiles of Quadratic Forms of Gaussian Vectors

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Abstract

We derive results on the asymptotic behavior of tails and quantiles of quadratic forms of Gaussian vectors. They appear in particular in delta-gamma models in financial risk management approximating portfolio returns. Quantile estimation corresponds to the estimation of the Value-at-Risk, which is a serious problem in high dimension.

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1 Introduction

Quadratic forms $X^\top QX$ of Gaussian vectors $X \sim N(\mu, \Sigma)$ play an important role in probability theory and statistics. These forms appear in (central and non-central) χ^2-statistics, likelihood ratios, and power spectra, which are used in many different applications and models throughout statistics.

Traditional applications include in engineering "ballistic analysis of multiple weapon systems", the "detection of signals from noise in multichannel receivers", "the study of bone lengths determined in vivo using X-ray stereography" (Jensen and Solomon, 1994) as well as numerous applications in communication theory cited by Raphaeli (1996) and Gao and Smith (1998).

This paper was triggered by a problem from financial mathematics. The so-called delta-gamma method approximates the Value-at-Risk, which is nothing else but a small quantile, e.g. the 1%-quantile. The approximation is based on a second order Taylor expansion of the price of a derivative, for instance, a European option. The expansion is for the price of the derivative at a particular time and at a certain price level of the underlying, which may be an index or an asset price. See Duffie and Pan (1997) for details.

In a Gaussian framework the second order approximation leads to

$$V(X) = \theta + \Delta^\top X + \frac{1}{2} X^\top \Gamma X, \tag{1.1}$$

where X is an m-dimensional Gaussian vector with mean 0 and covariance matrix Σ, Δ is a vector in $\mathbb{R}^m$, and Γ is some symmetric $m \times m$-matrix. The Gaussian model is usually based on the central limit theorem. Such quadratic approximations are extremely popular in risk management for financial institutions (Mina and Ulmer, 1999).

Equation (1.1) can be brought into the diagonal form

$$V = \theta + \delta^\top Y + \frac{1}{2} Y^\top \Lambda Y = \theta + \sum_{j=1}^{m} (\delta_j Y_j + \frac{1}{2} \lambda_j Y_j^2), \tag{1.2}$$

where $Y = (Y_1, \ldots, Y_m)^\top$ is a standard normal vector, $\delta = (\delta_1, \ldots, \delta_m)^\top \in \mathbb{R}^m$, and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ is a diagonal matrix. This can be done by solving the *generalized eigenvalue problem*

$$CC^\top = \Sigma,$$
$$C^\top \Gamma C = \Lambda,$$

and putting $X = CY$, $\delta = C^\top \Delta$.

Approximations to the probability distribution of V include series expansions (Mathai and Provost, 1992, section 4.2), numerical Fourier inversion (Rice, 1980), Monte-Carlo simulation (Glassermann et al., 2001), and numerous approximations with limited accuracy based on moment matching (see (Jaschke, 2001) for references). The two practically used approaches that in principle can achieve any desired accuracy are numerical Fourier inversion and Monte-Carlo simulation.

For small quantiles special Monte Carlo simulation methods, such as importance sampling (see e.g. (Glassermann et al., 2001)), have been developed to reduce the required amount of simulations.

The Fourier inversion method starts with the characteristic function $\phi(t) = Ee^{itV}$, $t \in \mathbb{R}$, which is known analytically in the case (1.1). Then the inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt \tag{1.3}$$

holds for the probability density f. The key to an error analysis of trapezoidal, equidistant approximations to the integral (1.3)

$$\tilde{f}(x, \Delta_t) := \frac{\Delta_t}{2\pi} \sum_{k=-\infty}^{\infty} \phi(k\Delta_t) e^{-ik\Delta_t x} \tag{1.4}$$

is the Poisson summation formula

$$\tilde{f}(x, \Delta_t, t) = \sum_{j=-\infty}^{\infty} f(x + \frac{2\pi}{\Delta_t} j). \tag{1.5}$$

The infinite sum has to be truncated, so the resulting errors consist in a discretization and a truncation error. The influence of these errors on accuracy is investigated theoretically as well as numerically in (Abate and Whitt, 1992, p.22).

This paper offers a different approach to the problem in providing an asymptotic approximation to the density f, its distribution tail F and the α–quantile x_α for α close to 0 or 1. This approach is in the spirit of Beran (1975), who derives the asymptotic right tail behavior of the distribution of V and its density function in the positive definite case, i.e. when $\lambda_1, \ldots, \lambda_m$ appearing in (1.2) are all strictly positive. The contribution of this paper is to develop the asymptotic tail behavior for the general case (1.2), without any restrictions on the $\lambda_i' s$. The extension of Beran's result was motivated by a real life example in risk management as mentioned before.

Our paper is organized as follows. In Section 2 we present V as being essentially a sum of independent non-central χ^2-distributed random variables with different degrees of freedom and non-centrality parameters. In Section 3 the main asymptotic results are derived. The behavior of the lower and upper tails of V are obtained for the relevant regimes, which are determined by the lowest/highest eigenvalue being negative, zero or positive. Section 4 is devoted to quantile approximation based on the results in Section 3. Examples and some discussion on our results in Section 5 conclude the paper.

2 An alternative representation

Suppose that the (generalized) eigenvalues λ_i of V appearing in (1.2) are sorted in increasing order. Suppose there are $n \leq m$ distinct eigenvalues, and denote by i_j the highest index of the j-th distinct eigenvalue, and by μ_j its multiplicity ($\mu_j = i_j - i_{j-1}, i_0 = 0, i_n = m$); thus it holds $\lambda_{i_1} < \ldots < \lambda_{i_n}$. For $j = 1, \ldots, n$, define

$$V_j := \begin{cases} \frac{1}{2}\lambda_{i_j} \sum_{l=i_{j-1}+1}^{i_j} \left(\frac{\delta_l}{\lambda_{i_j}} + Y_l \right)^2, & \text{if } \lambda_{i_j} \neq 0, \\ \sum_{l=i_{j-1}+1}^{i_j} \delta_l Y_l, & \text{if } \lambda_{i_j} = 0, \end{cases} \tag{2.1}$$

and $\bar{\delta}_j^2 := \sum_{l=i_{j-1}+1}^{i_j} \delta_l^2$. Then the V_j are independent and

$$V = \theta - \sum_{\substack{j=1 \\ \lambda_{i_j} \neq 0}}^{n} \frac{\bar{\delta}_j^2}{2\lambda_{i_j}} + \sum_{j=1}^{n} V_j.$$

If $\lambda_{i_j} = 0$, then V_j is Gaussian. If $\lambda_{i_j} \neq 0$, then V_j is a scaled version of a (non-central) χ^2-variable with μ_j degrees of freedom and non-centrality parameter $a_j^2 = \bar{\delta}_j^2 / \lambda_{i_j}^2$. Specifically, if $g(\cdot; a_j^2, \mu_j)$ denotes the $\chi^2_{\mu_j}(a_j^2)$-density, then

$$f_j(x) = \frac{2}{|\lambda_{i_j}|} g(\frac{2}{\lambda_{i_j}} x;\, a_j^2, \mu_j), \tag{2.2}$$

where f_j denotes the density of V_j.

3 Approximation of the tails

In this section we shall determine the tail behavior of the density $f(x)$ of V as x approaches the left and right endpoints of its support. It will turn out that the left, resp. right, tail behavior of f differs according whether λ_{i_1}, resp. λ_{i_n}, is negative, zero, or positive. For $\lambda_{i_1} < 0$, $f(x)$ behaves like a constant times $f_1(x)$ as $x \to -\infty$, for $\lambda_{i_1} \geq 0$ like a constant times a power of x times $f_1(x)$, as x approaches the left endpoint.

3.1 Case 1: The lowest eigenvalue is negative

For our results we shall need the tail behavior of (non-)central χ^2-distributions, whose density is known analytically, see for example (Johnson et al., 1994, p.416) and (Johnson et al., 1995, p.436):

$$g(x; a^2, \mu) = \mathbf{1}_{(0,\infty)}(x) \begin{cases} \frac{1}{2}(\sqrt{x}/a)^{\mu/2-1} I_{\mu/2-1}(a\sqrt{x}) e^{-(x+a^2)/2} & (a \neq 0) \\ \frac{1}{2^{\mu/2}\Gamma(\mu/2)} x^{\mu/2-1} e^{-x/2} & (a = 0), \end{cases} \tag{3.1}$$

where $a := \sqrt{a^2}$ and

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{3.2}$$

is the modified Bessel function of the first kind.

The tail behavior of $I_\nu(x)$ for $x \to \infty$ is independent of ν, see e.g. (Abramowitz and Stegun, 1965, (9.7.1)):

$$I_\nu(x) = e^x (2\pi x)^{-1/2} (1 + O(1/x)), \quad x \to \infty, \tag{3.3}$$

which leads to

$$g(x; a^2, \mu) = (2\sqrt{2\pi})^{-1} a^{(1-\mu)/2}\, e^{-a^2/2}\, x^{(\mu-3)/4}\, e^{-x/2+a\sqrt{x}} (1 + O(1/\sqrt{x})), \quad x \to \infty, \tag{3.4}$$

in the case $a \neq 0$. Together with (2.2) this leads to the tail behavior of f_j (if $\lambda_{i_j} \neq 0$):

$$f_j(x) = f_j^t(x)\,(1 + O(1/\sqrt{|x|})), \quad x \to (\operatorname{sgn} \lambda_{i_j})\infty, \tag{3.5}$$

where f_j^t is defined by

$$f_j^t(x) := c_j \mathbf{1}_{(0,\infty)}(\lambda_{i_j} x) \begin{cases} |x|^{(\mu_j-3)/4} e^{-x/\lambda_{i_j} + a_j \sqrt{|2/\lambda_{i_j}|}\sqrt{|x|}} & (a_j \neq 0) \\ |x|^{\mu_j/2-1}\ e^{-x/\lambda_{i_j}} & (a_j = 0), \end{cases} \tag{3.6}$$

with

$$c_j := \begin{cases} (2\sqrt{2\pi})^{-1} e^{-a_j^2/2} a_j^{(1-\mu_j)/2} \left(\frac{2}{|\lambda_{i_j}|}\right)^{(\mu_j+1)/4} & (a_j \neq 0) \\ |\lambda_{i_j}|^{-\mu_j/2} / \Gamma(\mu_j/2) & (a_j = 0) \end{cases}$$

and $a_j = \sqrt{a_j^2} = |\bar{\delta}_j/\lambda_{i_j}|$. Note also that the support of f_j is $[0,\infty)$ if $\lambda_{i_j} > 0$, and $(-\infty, 0]$ if $\lambda_{i_j} < 0$. If $\lambda_{i_j} = 0$, then f_j is Gaussian. The following theorem shows that the left tail behavior of f is determined by the tail behavior of f_1.

Theorem 3.1 *For $\lambda_1 = \lambda_{i_1} < 0$, the density f of V has the asymptotic left tail behavior*

$$f(x) = b_1 f_1(x)\,(1 + O(1/\sqrt{|x|})) = b_1 f_1^t(x)\,(1 + O(1/\sqrt{|x|})), \quad x \to -\infty, \tag{3.7}$$

and for $\lambda_m = \lambda_{i_n} > 0$, it has the asymptotic right tail behavior

$$f(x) = b_n f_n(x)\,(1 + O(1/\sqrt{x})) = b_n f_n^t(x)\,(1 + O(1/\sqrt{x})), \qquad x \to \infty, \tag{3.8}$$

where the constant b_k, $k \in \{1, n\}$, is given by

$$b_k := e^{\theta/\lambda_{i_k} - a_k^2/2} \prod_{j \in \{1,\dots,n\}\setminus\{k\}} \left(\left(1 - \frac{\lambda_{i_j}}{\lambda_{i_k}}\right)^{-\mu_j/2} e^{\bar{\delta}_j^2 (2(\lambda_{i_k} - \lambda_{i_j})\lambda_{i_k})^{-1}} \right). \tag{3.9}$$

Proof. Let $\lambda_1 < 0$. Our proof is inspired by an Example given in (Balkema et al., 1993, p. 573). We claim that, whenever a probability density h has asymptotic behavior $h(x) = c_h f_1^t(x)\,(1 + O(1/\sqrt{|x|}))$ (for some constant $c_h \neq 0$ and $x \to -\infty$), then, for $j > 1$, the convolution $h * f_j$ has asymptotic behavior

$$(h * f_j)(x) = \left(\int_{-\infty}^{\infty} e^{y/\lambda_1} f_j(y)\,dy \right) h(x)\,(1 + O(1/\sqrt{|x|})), \quad x \to -\infty. \tag{3.10}$$

In other words, we show that

$$\int_{-\infty}^{\infty} \sqrt{|x|} \left(\frac{h(x-y)}{h(x)} - e^{y/\lambda_1} \right) f_j(y) dy = O(1), \quad x \to -\infty. \tag{3.11}$$

To show (3.11), split the appearing integral into the two integrals ranging over $(-\infty, x+c)$ and $[x+c, \infty)$, for some sufficiently large positive constant c. The first integral can be bounded as

$$\int_{-\infty}^{x+c} \sqrt{|x|} \left| \frac{h(x-y)}{h(x)} - e^{y/\lambda_1} \right| f_j(y) dy \leq$$
$$\frac{\sqrt{|x|}}{h(x)} \Big(\sup_{-\infty < y \leq x+c} f_j(y) \Big) \int_{-\infty}^{x+c} h(x-y)\,dy + \sqrt{|x|} \int_{-\infty}^{x+c} e^{y/\lambda_1} f_j(y)\,dy.$$

The first term in this sum converges to 0 for $x \to -\infty$, since $\int_{-\infty}^{x+c} h(x-y)\,dy \leq 1$, and since $h(x)/\sqrt{|x|}$ decreases slower than f_j, see (3.5), (3.6). If we choose $\tilde{\lambda}_j \in (\lambda_1, \lambda_j)$, $\tilde{\lambda}_j < 0$, then (3.5) and (3.6) show $e^{y/\lambda_1} f_j(y) = O(e^{y(\lambda_1^{-1} - \tilde{\lambda}_j^{-1})})$, $y \to -\infty$, and thus $\int_{-\infty}^{x+c} e^{y/\lambda_1} f_j(y)\,dy = O(e^{(x+c)(\lambda_1^{-1} - \tilde{\lambda}_j^{-1})})$, $x \to -\infty$, showing that the second term above converges to 0 for $x \to -\infty$, too.

Now (3.11) will follow if we show that there is an integrable function majorizing

$$G : y \mapsto f_j(y) e^{y/\lambda_1} \sup_{x \leq x_0} \left\{ \sqrt{|x|} \left| \frac{h(x-y)}{h(x)} e^{-y/\lambda_1} - 1 \right| \mathbf{1}_{[x+c,\infty)}(y) \right\} \tag{3.12}$$

for some suitably chosen $x_0 < 0$. We will choose $x_0 = -2c$. Thus for the treatment of G we can assume $x \leq -2c$ and $x - y \leq -c$ in the following calculations. Write $h(x) = c_h f_1^t(x)\,(1 + \rho(x))$, where

$$|\sqrt{|x|}\,\rho(x)| \leq C \quad \forall\, x \leq -c \tag{3.13}$$

(c sufficiently large, C some constant). Then we have from (3.6)

$$
\begin{aligned}
&\frac{h(x-y)}{h(x)} e^{-y/\lambda_1} - 1 \\
= \quad & \left|\frac{x-y}{x}\right|^{(\mu_1-3)/4} e^{a_1\sqrt{2/|\lambda_1|}(\sqrt{y-x}-\sqrt{|x|})} \frac{1+\rho(x-y)}{1+\rho(x)} - 1 \\
= \quad & \left\{ \left|\frac{x-y}{x}\right|^{(\mu_1-3)/4} e^{a_1\sqrt{2/|\lambda_1|}(\sqrt{y-x}-\sqrt{|x|})} - 1 \right. \\
& + \left|\frac{x-y}{x}\right|^{(\mu_1-3)/4} e^{a_1\sqrt{2/|\lambda_1|}(\sqrt{y-x}-\sqrt{|x|})} \rho(x-y) \\
& \left. -\rho(x) \right\} \frac{1}{1+\rho(x)} \\
=: \quad & \left\{ H_1(x,y) + H_2(x,y) + H_3(x,y) \right\} \frac{1}{1+\rho(x)},
\end{aligned}
\tag{3.14}
$$

where $H_i(x,y)$ is defined to be the summand appearing in the i'th row of the preceding sum. We claim that for any $\lambda' > 0$, there is a constant $C_{\lambda'} > 0$ such that

$$
\sqrt{|x|}\,|H_i(x,y)| \le C_{\lambda'}\, e^{\lambda'|y|}, \ \forall\, x \le -2c,\ x-y \le -c, \quad i = 1,2,3. \tag{3.15}
$$

For H_3 this is clear, since $\sqrt{|x|}\,|H_3(x,y)| \le C$ by (3.13). To show this for H_2, note that

$$
\sqrt{|x|}\,\rho(x-y) = \frac{\sqrt{x}}{\sqrt{|x-y|}} \sqrt{|x-y|}\,\rho(x-y) \le C \left|\frac{x-y}{x}\right|^{-1/2}
$$

by (3.13), and hence

$$
\sqrt{|x|}\,|H_2(x,y)| \le \left|\frac{x-y}{x}\right|^{r} e^{p(\sqrt{y-x}-\sqrt{|x|})},
$$

for some $r \in \mathbb{R}$ and $p := a_1\sqrt{2/|\lambda_1|}$. Now if $y \ge 0$, then $1 \le |(x-y)/x| \le 1 + y/(2c)$. If $y < 0$, and $x \le 2y$, then $1/2 \le |(x-y)/x| \le 2$. If $y < 0$ and $x \ge 2y$, then $c/(2|y|+c) \le |(x-y)/x| \le 2$. Thus, for all $x \le -2c$ and $x - y \le -c$ it holds

$$
\frac{c}{2|y|+2c} \le \left|\frac{x-y}{x}\right| \le 2 + \frac{|y|}{2c}. \tag{3.16}
$$

Together with

$$
\sup_{x\le -2c} e^{p(\sqrt{y-x}-\sqrt{|x|})} \le \sup_{x\le -2c} e^{p(\sqrt{|y|+|x|}-\sqrt{|x|})} \le e^{p\sqrt{|y|}}
$$

this implies the validity of (3.15) for H_2. Next we shall show that it also holds for H_1: An application of the mean value theorem to the function $y \mapsto (1-y/x)^{(\mu_1-3)/4}$ shows that

$$
\left(1-\frac{y}{x}\right)^{(\mu_1-3)/4} = 1 - \frac{\mu_1-3}{4}\left(1-\frac{\xi_1}{x}\right)^{(\mu_1-3)/4-1} \frac{y}{x} =: 1 + \psi_1(x,y), \tag{3.17}
$$

where ξ_1 is some number between 0 and y. Since $1-\xi_1/x$ lies between 1 and $1-y/x$, it follows from (3.16) and $x \le -2c$, that

$$\sqrt{|x|}\,|\psi_1(x,y)| \le C_1 \left(2+\frac{2|y|}{c}\right)^{C_1'} \tag{3.18}$$

for some constants C_1, C_1'. Applying the mean value theorem to the function $y \mapsto e^{p(\sqrt{y-x}-\sqrt{-x})}$, where $p = a_1\sqrt{2/|\lambda_1|}$, shows

$$e^{p(\sqrt{y-x}-\sqrt{-x})} = 1 + e^{p(\sqrt{\xi_2-x}-\sqrt{-x})}\,\frac{py}{2\sqrt{\xi_2-x}} =: 1+\psi_2(x,y), \tag{3.19}$$

where ξ_2 is some number between 0 and y. Now since

$$e^{p(\sqrt{\xi_2-x}-\sqrt{-x})} \le e^{(p\sqrt{|y|+|x|}-\sqrt{|x|})} \le e^{p\sqrt{|y|}},$$

and since $\sqrt{|x|/(\xi_2-x)} = (1-\xi_2/x)^{-1/2}$, where $1-\xi_2/x$ lies between 1 and $1-y/x$, it follows as for ψ_1 that

$$\sqrt{|x|}\,|\psi_2(x,y)| \le C_2 \left(2+\frac{2|y|}{c}\right)^{C_2'} e^{p\sqrt{|y|}}, \tag{3.20}$$

for some constants C_2 and C_2'. Now (3.14), (3.17) and (3.19) show

$$\sqrt{|x|}\,H_1(x,y) = \sqrt{|x|}(\psi_1(x,y)+\psi_2(x,y)+\psi_1(x,y)\psi_2(x,y)),$$

and (3.18), (3.20) then immediately imply the trueness of (3.15) for H_1.

Since $\lim_{x\to-\infty}\rho(x)=0$ it follows from (3.12), (3.14) and (3.15), that for any $\lambda'>0$ there is $C'_{\lambda'}>0$ such that $G(y) \le C'_{\lambda'} f_j(y) e^{y/\lambda_1} e^{\lambda'|y|}$. But it is clear that this is an integrable majorant for sufficiently small λ'. Hence we obtain (3.11) and (3.10). It then follows by induction and from (3.5) that the density of $\sum_{j=1}^n V_j$ has asymptotic behavior

$$\left(\prod_{j=2}^n \int_{-\infty}^{\infty} e^{y/\lambda_1} f_j(y)\,dy\right) f_1^t(x)\,(1+O(1/\sqrt{|x|})), \tag{3.21}$$

as $x\to-\infty$. It remains to calculate $\int_{-\infty}^{\infty} e^{y/\lambda_1} f_j(y)\,dy$: For $\lambda_{i_j}\ne 0$, (2.2) gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{y/\lambda_1} f_j(y)\,dy \\
&= \int_{-\infty}^{\infty} e^{x\lambda_{i_j}/(2\lambda_1)} g(x; a_j^2, \mu_j)\,dx \\
&= E\left(\exp\left\{\frac{\lambda_{i_j}}{2\lambda_1}\chi^2_{\mu_j}(a_j^2)\right\}\right) \\
&= e^{\bar\delta_j^2/(2\lambda_{i_j}\lambda_1)} \left(1-\frac{\lambda_{i_j}}{\lambda_1}\right)^{-\mu_j/2} e^{\bar\delta_j^2/(2(\lambda_1-\lambda_{i_j})\lambda_1)},
\end{aligned} \tag{3.22}$$

where we used that the moment generating function of $\chi^2_{\mu_j}(a_j^2)$ at $t \le 1/2$ is given by

$$E(\exp\{t\,\chi^2_{\mu_j}(a_j^2)\}) = (1-2t)^{-\mu_j/2}\exp(a_j^2 t(1-2t)^{-1}),$$

see e.g. (Johnson et al., 1995, p. 437). Similar calculations, using the moment generating function of the normal distribution, show that

$$\int_{-\infty}^{\infty} e^{y/\lambda_1} f_j(y)\,dy = e^{\overline{\delta}_j^2/(2\lambda_1^2)}$$

for $\lambda_{i_j} = 0 \neq \overline{\delta}_j^2$. Then it follows with b_1 as defined in (3.9), that the density of $\sum_{j=1}^n V_j$ has the asymptotic behavior $\exp\{-\theta/\lambda_1 + a_1^2/2 + \sum\limits_{\substack{j=2\\ \lambda_{i_j}\neq 0}}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}\lambda_1}\} b_1 f_1^t(x)\,(1+O(1/\sqrt{|x|}))$, as $x \to -\infty$. Thus, since $V = \theta - \sum\limits_{\substack{j=1\\ \lambda_{i_j}\neq 0}}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}} + \sum_{j=1}^n V_j$,

$$f(x) = \exp\{-\theta/\lambda_1 + \sum_{\substack{j=1\\ \lambda_{i_j}\neq 0}}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}\lambda_1}\} b_1 f_1^t(x - \theta + \sum_{\substack{j=1\\ \lambda_{i_j}\neq 0}}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}})\,(1+O(1/\sqrt{|x|})),$$

as $x \to -\infty$. Then it follows immediately that $\lim_{x\to-\infty} f(x)(b_1 f_1^t(x))^{-1} = 1$. Using more precise arguments, similar to the ones we used to show (3.15) for H_1, together with (3.5) then imply (3.7). The proof of (3.8) is similar. □

Remark 3.2 (a) By (3.22) and (3.9), for $k \in \{1, n\}$,

$$b_k = E(\exp\{\frac{1}{\lambda_{i_k}}(V - V_k)\}),$$

thus b_k is nothing else than the moment generating function of $V - V_k$ evaluated at the point $1/\lambda_{i_k}$.
(b) Equation (3.7) is trivially true if $\lambda_1 > 0$, since then the support of f as well as the support of f_1 are both bounded from the left.
(c) Equation (3.7) shows that there is a function ϱ and constants $c, C > 0$ such that $f(x) = b_1 f_1^t(x)(1 + \varrho(x))$ and $|\sqrt{|x|}\varrho(x)| \le C$ for all $x \le -c$. The constant C gives error bounds for the approximation. The proof presented here is actually constructive, i.e. explicit values for c and C could be derived by exact book keeping in the proof. Bounds for the needed starting constants in (3.3) can be found in Olver (1964), for example.
(d) Similar results have been derived in the context of distribution tails; see Goldie and Klüppelberg (1998) and references therein.

3.2 Case 2: The lowest eigenvalue is positive

Suppose that $\lambda_1 > 0$. In this subsection we shall derive the tail behavior of $f(x)$ as x approaches the left endpoint of its support: From (3.2) follows that the modified Bessel function of the first kind $I_\nu(x)$ behaves like $2^{-\nu}(\Gamma(\nu+1))^{-1}x^\nu(1+O(x^2))$ as $x \searrow 0$. Then (3.1) shows

$$g(x; a^2, \mu) = \frac{2^{-\mu/2}}{\Gamma(\mu/2)} e^{-a^2/2} x^{\mu/2-1}(1+O(x)), \quad x \searrow 0,$$

and with (2.2) we obtain for $j = 1, \ldots, n$,

$$f_j(x) = \frac{\lambda_{i_j}^{-\mu_j/2}}{\Gamma(\mu_j/2)} e^{-a_j^2/2} x^{\mu_j/2-1}(1+\psi_j(x)),$$

where ψ_j is a function for which there exist constants $d_j, D_j > 0$ such that $|\psi_j(x)| \leq D_j|x|$ for all $x \in [0, d_j]$. Then we obtain for $j, k \in \{1, \ldots, n\}$,

$$f_j * f_k(x) = \frac{\lambda_{i_j}^{-\mu_j/2}\lambda_{i_k}^{-\mu_k/2} e^{-(a_j^2+a_k^2)/2}}{\Gamma(\mu_j/2)\Gamma(\mu_k/2)} \int_0^x y^{\mu_j/2-1}(x-y)^{\mu_k/2-1}(1+\psi_j(y))(1+\psi_k(x-y))\,dy.$$

Now it holds

$$\begin{aligned}
&\int_0^x y^{\mu_j/2-1}(x-y)^{\mu_k/2-1}\,dy \\
&= x^{(\mu_j+\mu_k)/2-1}\int_0^1 z^{\mu_j/2-1}(1-z)^{\mu_k/2-1}\,dz \\
&= x^{(\mu_j+\mu_k)/2-1} B(\mu_j/2, \mu_k/2) \\
&= x^{(\mu_j+\mu_k)/2-1} \frac{\Gamma(\mu_j/2)\Gamma(\mu_k/2)}{\Gamma(\frac{\mu_j+\mu_k}{2})},
\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the Beta-function. For the remaining terms, similar calculations show e.g.

$$\left|\int_0^x y^{\mu_j/2-1}(x-y)^{\mu_k/2-1}\psi_j(y)\,dy\right| \leq D_j \frac{\Gamma(\mu_j/2+1)\Gamma(\mu_k/2)}{\Gamma(\frac{\mu_j+\mu_k}{2}+1)} x^{(\mu_j+\mu_k)/2}$$

for $x \in [0, d_j]$, implying

$$f_j * f_k(x) = \frac{\lambda_{i_j}^{-\mu_j/2}\lambda_{i_k}^{-\mu_k/2}}{\Gamma(\frac{\mu_j+\mu_k}{2})} e^{-(a_j^2+a_k^2)/2} x^{(\mu_j+\mu_k)/2-1}(1+O(x)), \quad x \searrow 0.$$

Now we immediately obtain the tail behavior of f:

Proposition 3.3 *For $\lambda_1 = \lambda_{i_1} > 0$, the density f of V has the asymptotic left tail behavior*

$$f(x + \theta - \sum_{j=1}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}}) = d\,|x|^{m/2-1}\,(1 + O(x)), \quad x \searrow 0 \tag{3.23}$$

with the constant

$$d = \frac{\prod_{j=1}^{n} |\lambda_{i_j}|^{-\mu_j/2}}{\Gamma(m/2)} e^{-\sum_{j=1}^{n} a_j^2/2}. \tag{3.24}$$

If $\lambda_m = \lambda_{i_n} < 0$, then (3.23) holds for the right tail as $x \nearrow 0$.

3.3 Case 3: The lowest eigenvalue is 0

Now suppose that $\lambda_1 = 0$ and $\overline{\delta}_1^2 \neq 0$. Since $V_1 = \sum_{l=1}^{i_1} \delta_l Y_l$ is normally distributed with mean zero and variance $\overline{\delta}_1^2$, it follows that $f_1(x) = (\sqrt{2\pi}|\overline{\delta}_1|)^{-1} e^{-x^2/(2\overline{\delta}_1^2)}$. We shall see that the left tail behavior of $f(x)$ is essentially determined by the tail behavior of f_1.

Proposition 3.4 *Let h be a probability density with support in $[0, \infty)$ such that*

$$h(x) = c_h x^\mu (1 + O(x)), \quad x \searrow 0 \tag{3.25}$$

for some $\mu > -1$ and some constant $c_h \neq 0$. Further, suppose that h is bounded on every interval $[\Delta, \infty)$ for every $\Delta > 0$. Then

$$(f_1 * h)(x) = c_h \frac{\Gamma(\mu + 1)(\overline{\delta}_1^2)^{\mu+1}}{\sqrt{2\pi}|\overline{\delta}_1|} |x|^{-(\mu+1)} e^{-x^2/(2\overline{\delta}_1^2)} (1 + O(1/|x|)), \quad x \to -\infty. \tag{3.26}$$

Proof. For simplicity we assume that $\overline{\delta}_1^2 = 1$. The proof for general $\overline{\delta}_1^2$ is similar or alternatively can be deduced by a simple dilation argument.
Note that (3.25) is equivalent to

$$h(x) = c_h x^\mu e^{x^2/2} (1 + O(x)), \quad x \searrow 0.$$

Write

$$h(x) = c_h x^\mu e^{x^2/2} (1 + \psi(x)),$$

where

$$|\psi(x)| \leq Dx \;\; \forall\, x \in [0, \Delta] \tag{3.27}$$

and $D, \Delta > 0$ are suitable constants. Also, let

$$h(x) \le E \;\; \forall\, x \in [\Delta, \infty) \tag{3.28}$$

for some $E > 0$. Then we have for negative x,

$$\begin{aligned}
(f_1 * h)(x) &= \int_0^\infty h(y) f_1(x-y)\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\Delta c_h y^\mu e^{y^2/2} e^{-(x-y)^2/2}\, dy \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\Delta c_h y^\mu e^{y^2/2} \psi(y) e^{-(x-y)^2/2}\, dy \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_\Delta^\infty h(y) e^{-(x-y)^2/2}\, dy \\
&= \frac{c_h}{\sqrt{2\pi}} e^{-x^2/2} |x|^{-(\mu+1)} \int_0^{\Delta|x|} z^\mu e^{-z}\, dz \\
&\quad + \frac{c_h}{\sqrt{2\pi}} e^{-x^2/2} |x|^{-(\mu+1)} \int_0^{\Delta|x|} z^\mu \psi(z/|x|) e^{-z}\, dz \\
&\quad + \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_\Delta^\infty h(y) e^{xy} e^{-y^2/2}\, dy.
\end{aligned}$$

Noting that $\Gamma(\mu+1) = \int_0^\infty z^\mu e^{-z}\, dz$, we obtain

$$\begin{aligned}
&\frac{(f_1 * h)(x) - c_h \Gamma(\mu+1) e^{-x^2/2} |x|^{-(\mu+1)}/\sqrt{2\pi}}{c_h \Gamma(\mu+1) e^{-x^2/2} |x|^{-(\mu+1)}/\sqrt{2\pi}} \\
= \;& \frac{-\int_{\Delta|x|}^\infty z^\mu e^{-z}\, dz}{\Gamma(\mu+1)} + \frac{\int_0^{\Delta|x|} z^\mu \psi(z/|x|) e^{-z}\, dz}{\Gamma(\mu+1)} + \frac{\int_\Delta^\infty h(y) e^{xy} e^{-y^2/2}\, dy}{c_h \Gamma(\mu+1) |x|^{-(\mu+1)}} \\
=: \;& A_1(x) + A_2(x) + A_3(x).
\end{aligned}$$

It remains to show that $|x|(A_1(x) + A_2(x) + A_3(x))$ is bounded as $x \to -\infty$: Since

$$|x| \int_{\Delta|x|}^\infty z^\mu e^{-z}\, dz \le \frac{1}{\Delta} \int_{\Delta|x|}^\infty z^{\mu+1} e^{-z}\, dz \to 0, \quad x \to -\infty,$$

we have $A_1(x) = O(1/x)$ as $x \to -\infty$. From (3.27) we obtain

$$|x| \int_0^{\Delta|x|} z^\mu \psi(z/|x|) e^{-z}\, dz \le D \int_0^{\Delta|x|} z^{\mu+1} e^{-z}\, dz \le D\, \Gamma(\mu+2),$$

showing $A_2(x) = O(1/x)$ as $x \to -\infty$. Finally, (3.28) gives

$$|x|^{\mu+2} \int_\Delta^\infty h(y) e^{xy} e^{-y^2/2}\, dy \le E|x|^{\mu+2} \int_\Delta^\infty e^{xy}\, dy = E|x|^{\mu+1} e^{\Delta x} \to 0, \quad x \to -\infty,$$

showing $A_3(x) = O(1/x)$ as $x \to -\infty$. This gives (3.26). □

Combining Propositions 3.3 and 3.4, we obtain the left tail behavior of f:

Theorem 3.5 *For $\lambda_1 = \lambda_{i_1} = 0$, the density f of V has the asymptotic left tail behavior*

$$f(x + \theta - \sum_{j=2}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}}) = \frac{e^{-\sum_{j=2}^{n} a_j^2/2}}{\sqrt{2\pi}|\overline{\delta}_1|} \left(\prod_{j=2}^{n} |\overline{\delta}_1^2/\lambda_{i_j}|^{\mu_j/2} \right) |x|^{-\sum_{j=2}^{n} \mu_j/2} e^{-x^2/(2\overline{\delta}_1^2)} (1 + O(1/x))$$

as $x \to -\infty$. For $\lambda_m = \lambda_{i_n} = 0$, the density of V has the asymptotic right tail behavior

$$f(x + \theta - \sum_{j=1}^{n-1} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}}) = \frac{e^{-\sum_{j=1}^{n-1} a_j^2/2}}{\sqrt{2\pi}|\overline{\delta}_n|} \left(\prod_{j=1}^{n-1} |\overline{\delta}_n^2/\lambda_{i_j}|^{\mu_j/2} \right) x^{-\sum_{j=1}^{n-1} \mu_j/2} e^{-x^2/(2\overline{\delta}_n^2)} (1 + O(1/x))$$

as $x \to \infty$.

4 Approximation of the quantiles

In this section we give an approximation of the α and $(1-\alpha)$-quantile of V as $\alpha \to 0$. As before, denote the density of V by f and its distribution function by F. The α-quantile of V will be denoted by x_α, thus

$$x_\alpha = F^{\leftarrow}(\alpha) := \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}, \quad \alpha \in (0, 1).$$

Since for $\lambda_{i_1} < 0$ Theorem 3.1 expressed the left tail behavior of f in terms of the tail behavior of f_1^t, it is natural to approximate x_α using the quantile of some suitable function $\tilde{F}_1^t$ where $\frac{d}{dx}\tilde{F}_1^t(x) = f_1^t(x)(1 + O(1/\sqrt{|x|}))$ as $x \to -\infty$. This is done in the following theorem. Note that there the function $\tilde{F}_1^t(x)$ is given explicitely and that its quantiles can be calculated easily numerically:

Theorem 4.1 *Suppose $\lambda_1 = \lambda_{i_1} < 0$, or $\lambda_m = \lambda_{i_n} > 0$, respectively. Define on the relevant range (i.e. for large negative x, or for large positive x, respectively)*

$$\begin{aligned} \tilde{F}_1^t(x) &:= |\lambda_1| f_1^t(x) = |\lambda_1| c_1 |x|^{(\mu_1 - 3)/4} e^{-x/\lambda_1 + a_1\sqrt{2/|\lambda_1|}\sqrt{|x|}}, \\ 1 - \tilde{F}_n^t(x) &:= |\lambda_{i_n}| f_n^t(x) = \lambda_{i_n} c_n x^{(\mu_n - 3)/4} e^{-x/\lambda_{i_n} + a_n\sqrt{2/\lambda_{i_n}}\sqrt{x}}, \end{aligned}$$

where f_1^t and f_n^t are given by (3.6). Let b_1 and b_n be defined as in (3.9), and denote the α-quantiles of $\tilde{F}_1^t$ and $1-\tilde{F}_n^t$ by $(\tilde{F}_1^t)^{\leftarrow}(\alpha)$ and $(1-\tilde{F}_n^t)^{\leftarrow}(\alpha)$, respectively. Then, as $\alpha \to 0$, the lower and upper quantiles of V satisfy the following asymptotic equations, respectively:

$$x_\alpha = \lambda_{i_1} \log b_1 + (\tilde{F}_1^t)^{\leftarrow}(\alpha) + O(1/\sqrt{|(\tilde{F}_1^t)^{\leftarrow}(\alpha)|}), \tag{4.1}$$

$$x_{1-\alpha} = \lambda_{i_n} \log b_n + (1 - \tilde{F}_n^t)^{\leftarrow}(\alpha) + O(1/\sqrt{(1 - \tilde{F}_n^t)^{\leftarrow}(\alpha)}). \tag{4.2}$$

Proof. Define the shifted random variable $V_{(sh)} := V - \lambda_{i_1} \log b_1$. Denote its density by $f_{(sh)}$, its distribution function by $F_{(sh)}$ and its α-quantile by $x_{\alpha,(sh)} = F_{(sh)}^{\leftarrow}(\alpha)$. Put $x_{\alpha,1} := (\tilde{F}_1^t)^{\leftarrow}(\alpha)$. Since

$$x_\alpha = x_{\alpha,(sh)} + \lambda_{i_1} \log b_1,$$

(4.1) is equivalent to

$$x_{\alpha,(sh)} - x_{\alpha,1} = O(1/\sqrt{|x_{\alpha,1}|}), \quad x_{\alpha,1} \to -\infty. \tag{4.3}$$

It remains to show (4.3): an application of Theorem 3.1 to $V_{(sh)}$ shows $f_{(sh)}(x) = b_{1,(sh)} f_1^t(x)(1+O(1/\sqrt{|x|}))$, where $b_{1,(sh)} = b_1 \exp\{(-\lambda_{i_1} \log b_1)/\lambda_{i_1}\} = 1$, i.e.

$$f_{(sh)}(x) = f_1^t(x)(1 + O(1/\sqrt{|x|})). \tag{4.4}$$

On the other hand, with $\tilde{f}_1^t(x) := \frac{d}{dx}\tilde{F}_1^t(x) = |\lambda_1| \frac{d}{dx} f_1^t(x)$ we also obtain

$$\tilde{f}_1^t(x) = f_1^t(x)(1 + O(1/\sqrt{|x|})).$$

Thus it holds

$$\tilde{f}_1^t(x) = f_{(sh)}(x)(1 + O(1/\sqrt{|x|})), \quad x \to -\infty,$$

i.e. there are positive constants $c, C > 0$ such that

$$|f_{(sh)}(x) - \tilde{f}_1^t(x)| \leq \frac{C}{\sqrt{|x|}} f_{(sh)}(x) \ \ \forall\, x \leq -c.$$

Choose c such that in addition $f_{(sh)}(x) > 0$ for all $x \leq -c$. Then $F_{(sh)}$ is strictly increasing on $(-\infty, -c)$ and hence $F_{(sh)}^{\leftarrow}(F_{(sh)}(x)) = x$ for all $x \leq -c$. Defining

$$r(x) := F_{(sh)}(x) - \tilde{F}_1^t(x), \tag{4.5}$$

it follows that

$$|r(x)| \leq \int_{-\infty}^{x} |f_{(sh)}(y) - \tilde{f}_1^t(y)|\, dy \leq \frac{C}{\sqrt{|x|}} F_{(sh)}(x) \ \ \forall\, x \leq -c. \tag{4.6}$$

Now let $0 < \alpha < 1$ such that $x_{\alpha,1} \leq -c$. Noting that

$$x_{\alpha,(sh)} - x_{\alpha,1} = F_{(sh)}^{\leftarrow}(F_{(sh)}(x_{\alpha,(sh)})) - F_{(sh)}^{\leftarrow}(F_{(sh)}(x_{\alpha,1})),$$

the mean value theorem implies the existence of some constant ξ between $F_{(sh)}(x_{\alpha,(sh)})$ and $F_{(sh)}(x_{\alpha,1})$ such that by (4.5) and (4.6),

$$\begin{aligned} |x_{\alpha,(sh)} - x_{\alpha,1}| &= |F_{(sh)}(x_{\alpha,(sh)}) - F_{(sh)}(x_{\alpha,1})| \cdot |(F_{(sh)}^{\leftarrow})'(\xi)| \\ &= |r(x_{\alpha,1})| \cdot |(F_{(sh)}^{\leftarrow})'(\xi)| \\ &\leq \frac{C}{\sqrt{|x_{\alpha,1}|}} \frac{F_{(sh)}(x_{\alpha,1})}{f_{(sh)}(F_{(sh)}^{\leftarrow}(\xi))} \\ &= \frac{C}{\sqrt{|x_{\alpha,1}|}} \frac{F_{(sh)}(x_{\alpha,1})}{\xi} \frac{F_{(sh)}(F_{(sh)}^{\leftarrow}(\xi))}{f_1^t(F_{(sh)}^{\leftarrow}(\xi))} \frac{f_1^t(F_{(sh)}^{\leftarrow}(\xi))}{f_{(sh)}(F_{(sh)}^{\leftarrow}(\xi))} \end{aligned} \tag{4.7}$$

Since $|\xi - F_{(sh)}(x_{\alpha,1})| \le |r(x_{\alpha,1})| \le CF_{(sh)}(x_{\alpha,1})/\sqrt{|x_{\alpha,1}|}$, it follows that $\xi \in [F_{(sh)}(x_{\alpha,1})(1 - C/\sqrt{|x_{\alpha,1}|}), F_{(sh)}(x_{\alpha,1})(1 + C/\sqrt{|x_{\alpha,1}|})]$, and hence $\lim_{x_{\alpha,1}\to-\infty} F_{(sh)}(x_{\alpha,1})/\xi = 1$. In particular, $\xi \to 0$ as $x_{\alpha,1} \to -\infty$, and thus $y := F^{\leftarrow}_{(sh)}(\xi) \to -\infty$ as $x_{\alpha,1} \to -\infty$. Since

$$\lim_{y\to-\infty} \frac{F'_{(sh)}(y)}{(f_1^t)'(y)} = -\lambda_1 \neq 0$$

by (4.4), l'Hospital's rule implies

$$\lim_{x_{\alpha,1}\to-\infty} \frac{F_{(sh)}(F^{\leftarrow}_{(sh)}(\xi))}{f_1^t(F^{\leftarrow}_{(sh)}(\xi))} = \lim_{y\to\infty} \frac{F_{(sh)}(y)}{f_1^t(y)} = -\lambda_1.$$

Also, by (4.4),

$$\lim_{x_{\alpha,1}\to-\infty} \frac{f_1^t(F^{\leftarrow}_{(sh)}(\xi))}{f_{(sh)}(F^{\leftarrow}_{(sh)}(\xi))} = \lim_{y\to-\infty} \frac{f_1^t(y)}{f_{(sh)}(y)} = 1.$$

Thus (4.7) implies (4.3) and hence (4.1). The proof of (4.2) is similar. □

Theorem 4.1 gives an approximation of x_α in terms of the α-quantile of some function $\tilde{F}_1^t(x)$. There, $\tilde{F}_1^t(x) = |\lambda_1| f_1^t(x)$ was chosen. However, the proof of Theorem 4.1 showed that any function $\tilde{F}_1^t$ could have been chosen, as long as

$$\frac{d}{dx}\tilde{F}_1^t(x) = f_1^t(x)(1 + O(1/\sqrt{|x|})), \quad x \to -\infty.$$

For example, one might choose

$$\tilde{F}_1^t(x) := \int_{-\infty}^{x} f_1(y)\, dy.$$

Then (2.2) implies

$$(\tilde{F}_1^t)^{\leftarrow}(\alpha) = \frac{\lambda_{i_1}}{2}\chi^2_{1-\alpha,\mu_1}(a_1^2),$$

where $\chi^2_{1-\alpha,\mu}(a^2)$ denotes the $(1-\alpha)$-quantile of the χ^2-distribution with μ degrees of freedom and non-centrality parameter a^2. Thus we obtain:

Corollary 4.2 *Suppose $\lambda_1 = \lambda_{i_1} < 0$, or $\lambda_m = \lambda_{i_n} > 0$, respectively. Then $\alpha \to 0$ is equivalent to $\chi^2_{1-\alpha,\mu_k}(a_k^2) \to \infty$ for $k \in \{1, n\}$, and as $\alpha \to 0$, the lower and upper quantiles of V satisfy the following asymptotic equations for $\lambda_{i_1} < 0$ and $\lambda_{i_n} > 0$, respectively:*

$$x_\alpha = \lambda_{i_1}\log b_1 + \frac{\lambda_{i_1}}{2}\chi^2_{1-\alpha,\mu_1}(a_1^2) + O(1/\sqrt{\chi^2_{1-\alpha,\mu_1}(a_1^2)}), \tag{4.8}$$

$$x_{1-\alpha} = \lambda_{i_n}\log b_n + \frac{\lambda_{i_n}}{2}\chi^2_{1-\alpha,\mu_n}(a_n^2) + O(1/\sqrt{\chi^2_{1-\alpha,\mu_n}(a_n^2)}). \tag{4.9}$$

Corollary 4.2 links the quantiles of V with the quantiles of non-central χ^2-distributions. The latter can be calculated with many software packages, such as **R**, *Electronic Tables* or *StaTable*, the latter two both reviewed in Boomsma and Molenaar (1994). The package *S-Plus* has a routine implemented to calculate the distribution function of a non-central χ^2-distribution. However, it does not compute the inverse of this function, i.e. the quantiles. Nevertheless, using a bisection method, the quantiles can be approximated numerically.

The following theorem gives an approximation of the quantiles of V for the case that the lowest eigenvalue is 0:

Theorem 4.3 *Suppose $\lambda_1 = \lambda_{i_1} = 0$, or $\lambda_m = \lambda_{i_n} = 0$, respectively. Define on the relevant range*

$$\begin{aligned} \tilde{F}_1^t(x) &:= \left(\frac{|\bar{\delta}_1|}{\sqrt{2\pi}} e^{-\sum_{j=2}^n a_j^2/2} \prod_{j=2}^n |\bar{\delta}_1^2/\lambda_{i_j}|^{\mu_j/2} \right) (-x)^{-1-\sum_{j=2}^n \mu_j/2} e^{-x^2/(2\bar{\delta}_1^2)}, \\ 1 - \tilde{F}_n^t(x) &:= \left(\frac{|\bar{\delta}_n|}{\sqrt{2\pi}} e^{-\sum_{j=1}^{n-1} a_j^2/2} \prod_{j=1}^{n-1} |\bar{\delta}_n^2/\lambda_{i_j}|^{\mu_j/2} \right) x^{-1-\sum_{j=1}^{n-1} \mu_j/2} e^{-x^2/(2\bar{\delta}_n^2)}. \end{aligned}$$

Denote by $(\tilde{F}_1^t)^{\leftarrow}(\alpha)$ and $(1-\tilde{F}_n^t)^{\leftarrow}(\alpha)$ the α-quantiles of $\tilde{F}_1^t$ and $1-\tilde{F}_n^t$, respectively. Then, as $\alpha \to 0$, the lower and upper quantiles of V satisfy the following asymptotic equations, respectively:

$$x_\alpha = \theta - \sum_{j=2}^n \frac{\bar{\delta}_j^2}{2\lambda_{i_j}} + (\tilde{F}_1^t)^{\leftarrow}(\alpha) + O(1/(\tilde{F}_1^t)^{\leftarrow}(\alpha)^2), \tag{4.10}$$

$$x_{1-\alpha} = \theta - \sum_{j=1}^{n-1} \frac{\bar{\delta}_j^2}{2\lambda_{i_j}} + (\tilde{F}_n^t)^{\leftarrow}(\alpha) + O(1/(1-\tilde{F}_n^t)^{\leftarrow}(\alpha)^2). \tag{4.11}$$

Proof. We only treat the case $\lambda_1 = 0$. The treatment of the upper tail for $\lambda_m = 0$ is similar. Since the proof is similar to the proof of Theorem 4.1, using Theorem 3.5 instead of Theorem 3.1, we only show how to modify that proof. Put $V_{(sh)} := V + \sum_{j=2}^n \frac{\bar{\delta}_j^2}{2\lambda_{i_j}} - \theta = \sum_{j=1}^n V_j$. Let $f_{(sh)}$ and $F_{(sh)}$ be the density and distribution function of $V_{(sh)}$, and $x_{\alpha,(sh)}$ the corresponding α-quantile. Define $\tilde{f}_1^t(x) := \frac{d}{dx}\tilde{F}_1^t(x)$ for $x < 0$. Then

$$\tilde{f}_1^t(x) = \frac{e^{-\sum_{j=2}^n a_j^2/2}}{\sqrt{2\pi}|\bar{\delta}_1|} \left(\prod_{j=2}^n |\bar{\delta}_1^2/\lambda_{i_j}|^{\mu_j/2} \right) |x|^{-\sum_{j=2}^n \mu_j/2} e^{-x^2/(2\bar{\delta}_1^2)} (1 + (1 + \sum_{j=2}^n \mu_j/2)\bar{\delta}_1^2 x^{-2}),$$

and Theorem 3.5 gives $\tilde{f}_1^t(x) = f_{(sh)}(x)(1+O(1/x))$. Then with $x_{\alpha,1} = (\tilde{F}_1^t)^{\leftarrow}(\alpha)$ denoting the α-quantile of $\tilde{F}_1^t$ and the same notations as in the proof of Theorem 4.1, (4.6) becomes

$|r(x)| \leq \frac{C}{|x|} F_{(sh)}(x)$, and (4.7) changes to

$$
\begin{aligned}
|x_{\alpha,(sh)} - x_{\alpha,1}| &\leq \frac{C}{|x_{\alpha,1}|} \frac{F_{(sh)}(x_{\alpha,1})}{\xi} \frac{F_{(sh)}(F^{\leftarrow}_{(sh)}(\xi))}{-\tilde{f}_1^t(F^{\leftarrow}_{(sh)}(\xi))(F^{\leftarrow}_{(sh)}(\xi))^{-1}} \frac{\tilde{f}_1^t(F^{\leftarrow}_{(sh)}(\xi))}{f_{(sh)}(F^{\leftarrow}_{(sh)}(\xi))} \frac{1}{-F^{\leftarrow}_{(sh)}(\xi)} \\
&= \frac{C'}{|x_{\alpha,1}|} \frac{1}{-F^{\leftarrow}_{(sh)}(\xi)} (1 + o(1)),
\end{aligned}
$$

where l'Hospital's rule was applied to $F_{(sh)}(F^{\leftarrow}_{(sh)}(\xi))/(-\tilde{f}_1^t(F^{\leftarrow}_{(sh)}(\xi))(F^{\leftarrow}_{(sh)}(\xi))^{-1})$. This implies (4.10). □

Finally, for $\lambda_{i_1} > 0$, an approximation can be written down quite explicitely, which is done in the next Theorem:

Theorem 4.4 *Suppose $\lambda_1 = \lambda_{i_1} > 0$, or $\lambda_m = \lambda_{i_n} < 0$, respectively. Then, as $\alpha \to 0$, the lower and upper quantiles of V satisfy the following asymptotic equations, respectively:*

$$
\begin{aligned}
x_\alpha &= \theta - \sum_{j=1}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}} + (\frac{m}{2d}\alpha)^{2/m} + O(\alpha^{4/m}), \\
x_{1-\alpha} &= \theta - \sum_{j=1}^{n} \frac{\overline{\delta}_j^2}{2\lambda_{i_j}} - (\frac{m}{2d}\alpha)^{2/m} + O(\alpha^{4/m}),
\end{aligned}
$$

where d is the constant defined in (3.24).

The proof is similar to the proof of Theorem 4.3 and therefore omitted.

5 Examples and discussion

In this section we shall illustrate the results of the last section at specific examples. The obtained approximations will be compared to standard approximation methods, like a normal approximation for $\lambda_1 \leq 0$, and a gamma approximation for $\lambda_1 > 0$.

Example 5.1 *(Illustration of Case 1)*
Suppose that in the model (1.2) we have $m = 15$, $n = 3$, $\lambda_{i_1} = -2$, $\lambda_{i_2} = 1$, $\lambda_{i_3} = 2$, $\mu_1 = 5$, $\mu_2 = 4$, $\mu_3 = 6$, $a_1^2 = 4$, $\overline{\delta}_2^2 = 4$, $\overline{\delta}_3^2 = 16$, and $\theta = 0$. In Figure 1 the left part of the distribution function of V, the normal approximation as well as the approximations according to Theorem 4.1 and Corollary 4.2 are plotted. The "true" distribution has

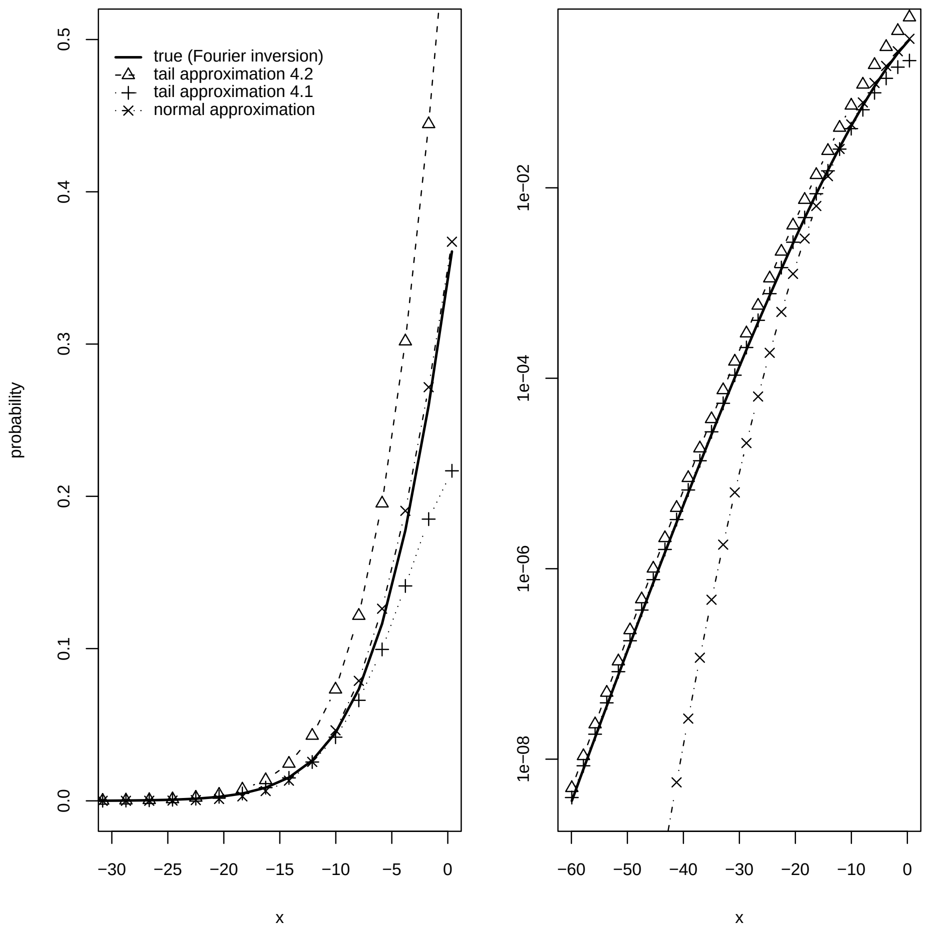

Figure 1: The left part of the distribution function (CDF) in the Example 5.1 (case 1: $\lambda_1 < 0$) as well as the normal approximation and the approximations of Theorem 4.1 and Corollary 4.2. The right graph shows probabilities in a log scale, the left in a linear scale.

been computed by numerical Fourier inversion with high accuracy. The left graph shows the probability on a linear scale, while the right graph shows it on a logarithmic scale. From the left graph it can be seen that the normal distribution approximates the true distribution well for small $|x|$, whereas the approximations of Theorem 4.1 and Corollary 4.2 approximate better for large negative x, which is shown by the right graph. The normal approximation is computed by moment matching: the cumulants of V can easily be read off the power series expansion of the cumulant generating function and are given by

$$\kappa_1 = \theta + \frac{1}{2}\sum_{j=1}^{m} \lambda_j \quad \text{and} \quad \kappa_r = \frac{1}{2}\sum_{j=1}^{m} \left((r-1)!\lambda_j^r + r!\delta_j^2 \lambda_j^{r-2}\right).$$

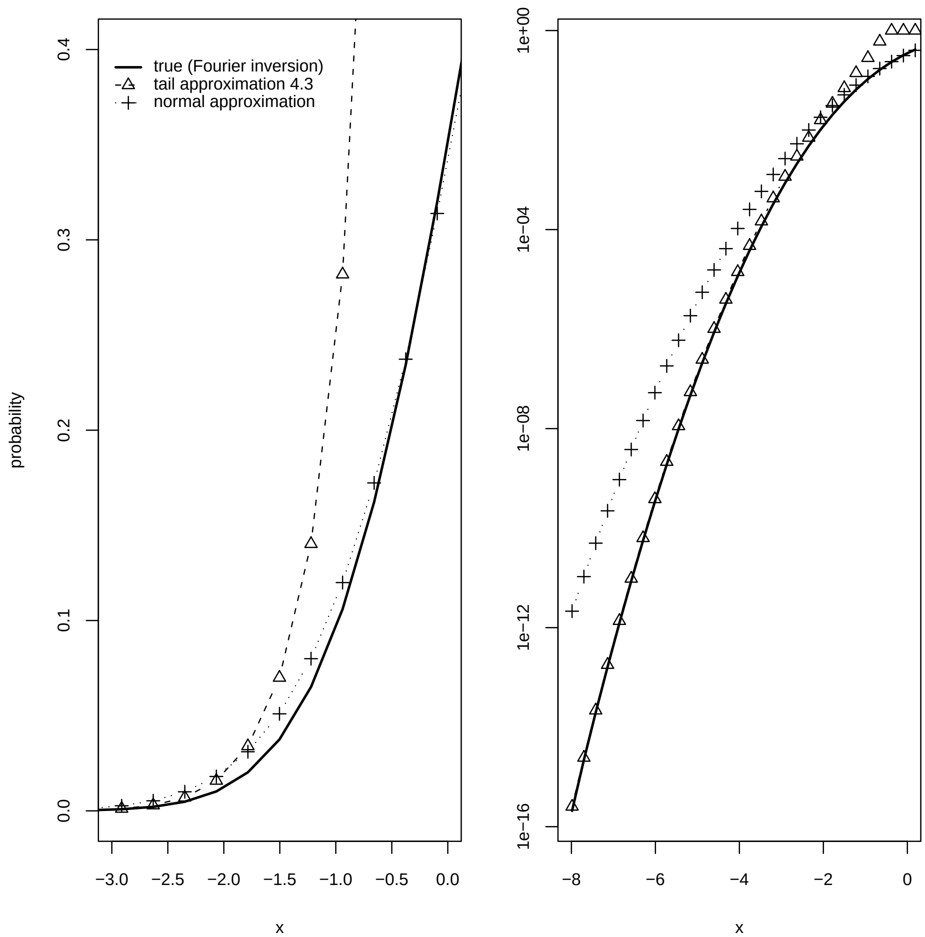

Figure 2: The left part of the distribution function (CDF) in the Example 5.2 (case 3: $\lambda_1 = 0$) as well as the normal approximation and the approximation of Theorem 4.3. The right graph shows probabilities on a logarithmic scale, the left one on a linear scale.

Example 5.2 *(Illustration of Case 3)*
Suppose that in the model (1.2) we have $m = n = 2$, $\lambda_1 = 0, \lambda_2 = 1$, $\delta_1 = 1$, $\delta_2 = 0$, and $\theta = 0$. Again, a normal approximation is quite good at the center of the distribution, whereas the approximation of Theorem 4.3 works well for large negative x. Table 1 shows that the tail approximation becomes better than the normal approximation for probabilities approximately below 0.025. In Figure 2, the distribution function of V, the normal approximation as well as the approximation of Theorem 4.3 are plotted on a linear and logarithmic scale.

Example 5.3 *(Illustration of Case 2)*
Suppose that in the model (1.2) we have $m = 4$, $n = 2$, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 2$,

probability	"true" quantile	normal approximation	tail approximation
0.0500	-1.3602	-1.514526	-1.636064
0.0250	-1.6916	-1.900456	-1.900803
0.0100	-2.0745	-2.349183	-2.228890
0.0050	-2.3339	-2.654734	-2.461087
0.0010	-2.8662	-3.284746	-2.954294
0.0001	-3.5131	-4.054846	-3.572531

Table 1: Quantiles in the example 5.2 (case 3: $\lambda_1 = 0$.)

$\delta_1 = \delta_2 = 1$, $\delta_3 = \delta_4 = 0$. $\theta = 1$ is chosen such that the left tail of the distribution ends at 0. A straightforward approximation of such a distribution is a gamma distribution (with shape parameter p and scale parameter β) with matching mean (βp) and variance ($\beta^2 p$). The gamma approximation fits very well at the center of the distribution, as seen from the left graph of figure 3, while the tail approximation of Theorem 4.4 is superior for $\alpha < 0.05$, approximately.

Remark 5.4 Since the tail approximations derived in the previous section are qualitatively different for $\lambda_1 < 0$, $\lambda_1 = 0$, and $\lambda_1 > 0$, it is clear that the approximation of Theorem 4.1 must give bad results for $\lambda_1 < 0$, but close to zero.

Example 5.5 This example shows that it can happen that (4.8) and (4.9) approximate well only for very small α: Let $n = m = 2$, $\mu_1 = \mu_2 = 1$, $-\lambda_1 = \lambda_2 = 2$, $\delta_1 = \delta_2 = 2a$, where $a \geq 3$ is positive, and $\theta = 0$. Then $V_1 = -(-a + Y_1)^2$, $V_2 = (a + Y_2)^2$, where Y_1 and Y_2 are independent standard normal variables. Then $P(Y_i \in [-3, 3]) \geq \sqrt{0.99}$ and it follows that $P(V_2 \in [(a-3)^2, (a+3)^2]) = P(V_1 \in [-(a+3)^2, -(a-3)^2] \geq \sqrt{0.99}$. Since V_1 and V_2 are independent, it follows $P(V_1 + V_2 \in [-12a, 12a]) \geq 0.99$, implying that the true 1%-quantile of $V_1 + V_2$ lies in $[-12a, 12a]$. However, if we use the approximation (4.8), we have $b_1 = 2^{-1/2} e^{-a^2/4}$, hence $\hat{x}_{1\%} = a^2/2 + \log 2 - \chi^2_{1-\alpha,1}(a^2)$, where $\hat{x}_{1\%}$ denotes the approximating quantity. Since the 1%-quantile of $\chi^2_1(a^2)$ lies in $[(a-2.6)^2, (a+2.6)^2]$, it follows that $\hat{x}_{1\%} \in [\log 2 - 6.76 - a^2/2 - 5.2a, \log 2 - 6.76 - a^2/2 + 5.2a]$. For large a, this differs clearly from the true quantile, which lies in $[-12a, 12a]$. So we see that the approximation (4.8) can lead to large errors in the approximation, if the level 1 % is fixed, and if the noncentrality-parameters are large, even if the eigenvalues λ_1 and λ_2 have the same modulus.

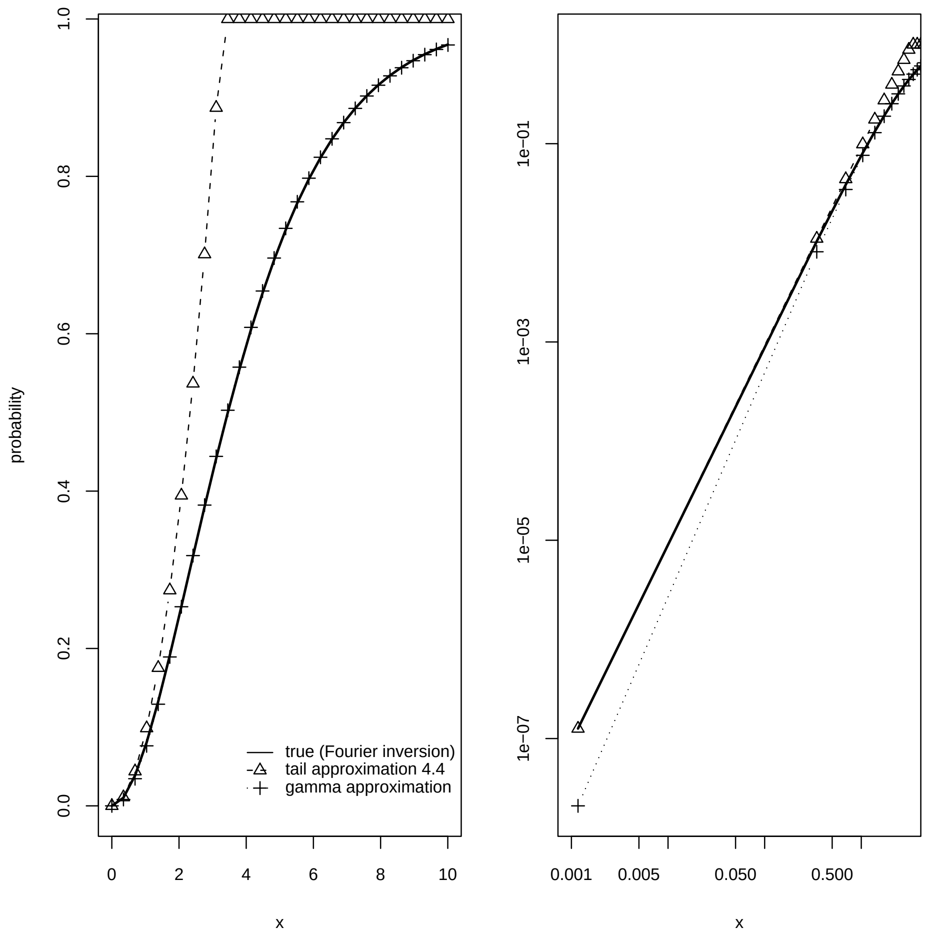

Figure 3: The left part of the distribution function (CDF) in the example 5.3 (case 2: $\lambda_1 > 0$), as well as the gamma approximation and the approximation of Theorem 4.4. The right graph shows x and the probabilities in a log scale, the left one in a linear scale.

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